Laboratory: Local Growth Rates of Polynomials

1 Before the Keyboard

Calculus is the language that we use to describe change. Here we will explore how some familiar functions change. We will look at polynomials and see how changes in their inputs result in changes in their outputs.

Example We start with the squaring function. We may write it as $f(x) = x^2$, or as $y = x^2$, or sometimes as $x \mapsto x^2$, read 'x maps to x^2'.

First consider how this function changes near to 3.

input x	*change in input* Δx	*output* y	*change in output* Δy
2.99	−0.01	8.9401	−0.0599
2.999	−0.001	8.994001	−0.005999
3	0	9	0
3.001	0.001	9.006001	0.006001
3.01	0.01	9.0601	0.0601

Note that as the function's input rises, so does its output. More than that, the output changes correspond with the input changes in a regular way – near to 3, the value of Δy is always about six times the value of Δx.

This lab studies the relationship between Δx and Δy for some polynomials.

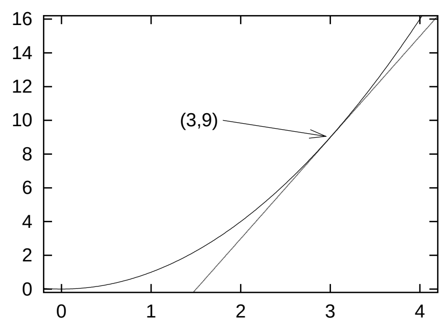
Example Continuing with the case of $f(x) = x^2$ near 3, we can calculate how the 6 arises: $(3 + \Delta x)^2 = (3 + \Delta x)(3 + \Delta x) = 9 + 6\Delta x + (\Delta x)^2$ and therefore $\Delta y = 6\Delta x + (\Delta x)^2$. Ths table shows why, of the two terms involving Δx in that expansion, the first-power one $6\Delta x$ plays a much larger role.

change in input Δx	*change in output* Δy	$6\Delta x$	$(\Delta x)^2$
−0.01	−0.0599	−0.06	0.0001
−0.001	−0.005999	−0.006	0.000001
0	0	0	0
0.001	0.006001	0.006	0.000001
0.01	0.0601	0.06	0.0001

The numbers in the $(\Delta x)^2$ column are much smaller than the numbers in the $6\Delta x$ column. We can easily see why this is true: for instance, if $\Delta x = 0.01$ then $(\Delta x)^2 = \Delta x \cdot \Delta x = 0.01 \cdot \Delta x$ is smaller than Δx by two orders of magnitude, since $0.01 = 10^{-2}$. Squaring a small number exaggerates its smallness.

We shall use a for the fixed point, and so we can summarize the above work with: near $a = 3$, that is, for small Δx, instead of working with the exact change relationship $3 + \Delta x \mapsto 9 + \Delta y$ we can work with the approximate relationship $3 + \Delta x \mapsto 9 + 6\Delta x$. This approximation is good enough for many purposes; for instance, if $\Delta x = 0.004$ then $f(3 + 0.004) = 9.024016$ so the exact Δy is 0.024016 while the approximation gives $6\Delta x = 6 \cdot 0.004 = 0.024$.

First-power relationships graph as lines, so $3 + \Delta x \mapsto 9 + 6\Delta x$ is a line with slope six passing through $(3, 9)$. The relationship that we are studying here is called the *linear approximation*, or *first-power appproximation* or more loosely *first approximation*, or sometimes the *tangent-line approximation* (because of the obvious connection shown on the graph below with tangent line work from plane geometry).



Near to the point $(3, 6)$ the line is near to the curve; that's what we mean by saying that for small Δx the approximation is good.

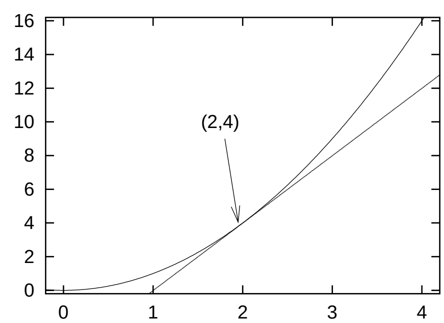
Example We can do a similar analysis of the behavior of $f(x) = x^2$ near $a = 2$.

input x	*change in input* Δx	*output* y	*change in output* Δy
1.99	−0.01	3.9601	−0.0399
1.999	−0.001	3.996001	−0.003999
2	0	4	0
2.001	0.001	4.004001	0.004001
2.01	0.01	4.0401	0.0401

Here, $\Delta y \approx 4\Delta x$. The expansion $(2 + \Delta x)^2 = 4 + 4\Delta x + (\Delta x)^2$ and this table

change in input Δx	*change in output* Δy	$4\Delta x$	$(\Delta x)^2$
-0.01	-0.0399	-0.04	0.0001
-0.001	-0.003999	-0.004	0.000001
0	0	0	0
0.001	0.004001	0.004	0.000001
0.01	0.0401	0.04	0.0001

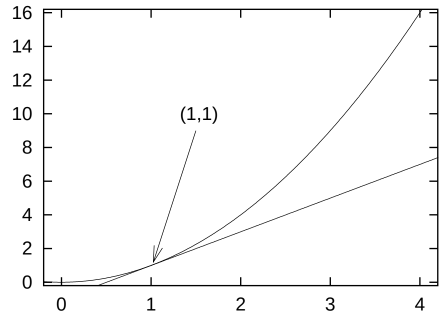
combine to explain this Δx-to-Δy relationship, As in the prior example, the second-power term Δx^2 contributes much less to the Δy than does the first-power term $4\Delta x$. Therefore, we can model the behavior of $f(x) = x^2$ near $a = 2$ with $\Delta y = 4\Delta x$.



Example Consider again the local behavior of $f(x) = x^2$, this time near $a = 1$. Because $(1 + \Delta x)^2 = 1 + 2\Delta x + (\Delta x)^2$ we have this table (which simply combines the pair of tables from each of the examples above).

input x	*change in input* Δx	*output* y	*change in output* Δy	$2\Delta x$	$(\Delta x)^2$
0.99	-0.01	0.9801	-0.0199	-0.02	0.0001
0.999	-0.001	0.998001	-0.001999	-0.002	0.000001
1	0	1	0	0	0
1.001	0.001	1.002001	0.002001	0.002	0.000001
1.01	0.01	1.0201	0.0201	0.02	0.0001

Again, we form the linear approximation by discounting the contribution of higher-order terms and get:



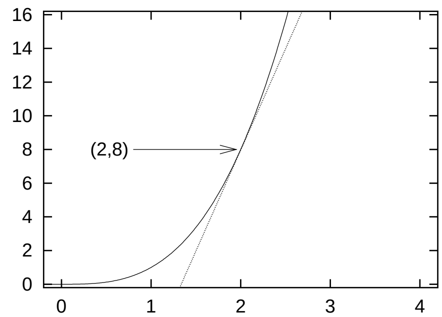
near to $(a, a^2) = (1, 1)$ the function's action is approximately equal to that of the line with slope $\Delta y/\Delta x = 2$ which is tangent to the curve at that point.

In general, for small Δx, an expression involving a higher power of Δx gives much smaller values than does an expression involving the first power of Δx. Note that this is the opposite of our experience for numbers much larger than zero, where a square is much bigger than a first-power term.

Example Next, consider the cubing function, written $y = x^3$, or $f(x) = x^3$, or $x \mapsto x^3$. We will describe how this function changes near $a = 2$. Expanding the expression $(2 + \Delta x)^3 = 8 + 12\Delta x + 6(\Delta x)^2 + (\Delta x)^3$ suggests this table.

input x	*change in input* Δx	*output* y	*change in output* Δy	$12\Delta x$	$6\Delta x^2$	Δx^3
1.99	−0.01	7.880599	0.119401	−0.12	0.0006	0.000001
1.999	−0.001	7.988006	0.011994	−0.012	0.000006	−0.000000001
2	0	8	1	0	0	0
2.001	0.001	8.012006	0.012006	0.012	0.000006	0.000000001
2.01	0.01	8.120601	0.120601	0.12	0.0006	0.000001

As earlier, the numbers in the higher-power change columns $6\Delta x^2$ and Δx^3 are much smaller than the numbers in the first-power change column $12\Delta x$. Consequently, near $a = 2$ we can get a good approximation by focusing on the linear change relationship $\Delta y \approx 12\Delta x$. Graphically, because $\Delta y/\Delta x \approx 12$ near to $(2, 8)$, the function changes approximately as does the line with slope 12 that is tangent to the curve at that point.



In this prepatory section we have considered how some familiar functions change near to a few points. The next section continues this exploration with higher-powered polynomials, using the computer to do some of the larger calculations.

Exercises

1. Consider the squaring function at $a = 3$. Find the change in y divided by the change in x when $x = 5$. Do the same when $x = 4$, and note that they are different. *Remark.* This exercise shows that it is impossible in general to assign a 'slope' number to a curve that is not a line. Instead, to describe the rate of change of a curve we must do something like we did above — associate a line with the curve and use the slope of that line.

2. For the squaring function examples above, give the $y = mx + b$ form of the equation of the tangent line in the $a = 3$, $a = 2$, and $a = 1$ cases.

3. For the squaring function, give the linear approximation at $a = -3$, $a = -2$, and $a = -1$.

4. Linear approximations are commonly used in science and engineering, as this passage from *Surely You're Joking, Mr. Feynman* illustrates. (This book is the autobiography of the Nobel prize winning physicist, and curious character, Richard Feynman. Here he is talking about his Ph.D. thesis advisor.)

> When I was at Los Alamos I found out that Hans Bethe was absolutely top-notch at calculating. For example, one timewe were putting some numbers into a formula and got 48 squared. I reach for the Marchant calculator, and he says, "That's 2300." I begin to push the buttons , and he says, "If you want it exactly, it's 2304."
>
> The machine says 2304, "Gee! That's pretty remarkable!" I say.
>
> "Don't you know how to square numbers near 50?" he says "You square 500 – that's 2500 – and subtract 100 times the difference of your number from 50 (in this case it's 2), so you have 2300. If you want the correction, square the difference and add it on. That makes 2304.

Derive Bethe's formula, both the exact one and the approximate one.

5. For the cubing function, give the equation of the tangent line in the $a = 2$ case. Then give the equation in the $a = 1$ case (for he first part you can use the expansion of $(2 + \Delta x)^3$ that is found above, but for this part you must expand $(1 + \Delta x)^3$ yourself).
6. Above it says 'for small Δx, an expression involving a higher power of Δx gives much smaller values than does an expression involving the first power of Δx'. There a subtlety in that assertion. Make a table with columns headed $(\Delta x)^2$ and $0.0001\Delta x$. Fill in rows where Δx has the value 0.1 and 0.01. Note that the first-power expression is not smaller than the second-power expression (because of the 0.0001 coefficient). Next, fill in rows where Δx is 0.001, 0.0001, and 0.00001. Note the first-power expression is now smaller than the second-power expression. The discrepancy will only get greater for smaller and smaller Δx's. That is, after we've seen this example, we see that the sentence above may be altered to say 'for *sufficiently* small Δx, an expression involving a higher power of Δx gives much smaller values than does an expression involving the first power of Δx'.
7. Give the formula for the slope of the line tangent to the squaring function at a. That is, expand the expression $(a + \Delta x)^2$ and discard terms involving higher power of Δx to get that Δy is approximately equal to an expression involving a times Δx.
8. We have focused on the behavior of the functions near to the point a. But the exact equations $a + \Delta x \mapsto f(a + \Delta x)$ remain true even when Δx is large. Consider again the relationship for $a = 3$ and the squaring function $3+\Delta x \mapsto 9+6\Delta x+(\Delta x)^2$. Make a table with columns headed x, Δx, y, Δy, $6\Delta x$, and Δx^2. Fill in the rows for $x = 3$, $x = 4$, $x = 5$, $x = 10$, and $x = 30$. Also fill in rows for $x = 2$, $x = 0$, and $x = -30$. Compute how far off the linear approximation is in each case. (So, when Δx is not small, we cannot discount the contribution from the quadratic term. On the contrary, those contributions come to dominate the expression $9+6\Delta x+(\Delta x)^2$.)

2 At the Keyboard

Now we will get the computer to carry out a similar analysis, in cases that are awkward for by-hand work. (*Remark:* a result called the Binomial Theorem would give us the conclusions also. But, rather than introduce and prove that theorem, we will get the numbers that we want by brute-force calculation. This is not elegent, but it does get us quickly into how functions change.)

Experiment Again consider the squaring function in the $a = 3$ case.

```
> expand((3+Delta_x)^2);
                     2
9+6*Delta_x+Delta_x
```

Maple can graph both the function and its linear approximation.

```
> plot({(3+Delta_x)^2),9+6*Delta_x},Delta_x=-2..2);
```

This plot is shifted from the one shown in the prior section because it takes Δx as the variable whereas above the variable is x. The first exercise asks you to get Maple to produce the original plot.

We can of course move beyond reproducing prior results.

Experiment Consider $f(x) = x^4$ and $a = 1$.

```
> expand((1+Delta_x)^4);
                          2           3          4
1+4*Delta_x+6*Delta_x +4*Delta_x +Delta_x
```

Discarding the terms with higher powers of Δx gives the first-power approximation $\Delta y = 1 + 4\Delta x$. Here is the plot (as in the prior example, this plot is shifted).

```
> plot({(1+Delta_x)^4,1+4*Delta_x},Delta_x=-2..2);
```

Note that this first computer case shws the same effect as the by-hand cases: discarding higher-power terms gives a good approximation for small Δx.

Experiment To emphasize some of the advantage that the computer has for these experiments, we can do the $a = 3$ case with the function $f(x) = x^{10}$.

```
> expand((1+Delta_x)^10);}
                                    2                3
59049+196830Delta_x+29524Delta_x +262440Delta_x + ...
> plot({(3+Delta_x)^10,59049+196830*Delta_x},Delta_x=-0.5..0.5);
```

Even quite-high power functions are not too much for the computer.

```
> expand((5+Delta_x)^20);

95367431640625+381469726562500Delta_x+ ...
>plot({(5+Delta_x)^20,95367431640625+381469726562500*Delta_x},Delta_x=-0.2..0.2);
```

The higher powers show the same effect, that the linear approximation is close to the curve for small Δx.

We can ask the computer to find what is the pattern; how is the coefficient of the first power of Δx determined from the power n in $f(x) = x^n$ and the value a?

Experiment Consider the case of the cubing function, but this time we do not give the computer a specific a.

```
> expand((a+Delta_x)^3);
 3   2                  2       3
a +3a Delta_x+3aDelat_x +Delta_x
```

Thus, in general, the formula for the linear approximation is $a + \delta_x \mapsto a^3 + 3a^2\Delta x$, that is, the line tangent to $f(x) = x^3$ at (a, a^3) has slope $3a^2$.

Experiment The fourth power function gives this.

```
> expand((a+Delta_x)^4);
 4   3           2       2           3        4
a +4a Delta_x+6a Delta_x +4aDelta_x +Delta_x
```

Thus the linear approximation is $a + \Delta x \mapsto a^4 + 4a^3\Delta x$, and so the line tangent to the graph at (a, a^4) has slope $4a^3$.

Those examples show an obvious similarity. The second exercise carries on in the same direction.

These experiments show the strength, and the weakness, of the computer in mathematical investigations. Oe would not be keen to expand the tenth power or hundredth power by hand. On the other hand, no number of examples constitute an ironclad proof. On balance, the point is that the computer has become invaluable for experimentation, conjecture, and trial.

Exercises

1. Which commands will shift the plot so they look like the ones in the prior section?
2. Make a table with columns labelled 'power n' and 'coefficient of Δx in the expansion of $(1 + \Delta x)^n$. Fill out the row for the powers $n = 2$, 3, 4, 5, 10, and 20. What seems to be the pattern?
3. Repeat the prior exercise for $(2 + \Delta x)^n$. Repeat also for $(-1 + \Delta x)^n$.
4. Conjecture as to the formula for a general power n at the point a.
5. On the basis of the prior exercise, guess at the equation of the line tangent to the graph of the square root function $f(x) = x^{1/2}$ at the point $(9, 3)$. Graph the function and your guessed tangent. Does you guess appear correct?
6. Briefly state the strength of evidence that you have for your conjecture. Is it a proof? How could it be made better, short of a proof?

3 After the Keyboard

Certainly the first question brought out by the material here is, "Why? Why look at approximations when we can consider the exact function, especially when we have the computer to do any hard calculations?" Exercise 1 below shows how the things studied above allow us to find answers that we do not have any evident way to find otherwise. Besides, the question of the exact formula for the linear approximation is interesting in and of itself.

With that motivation, we can move forward in two ways. First, we can try to prove that the patterns seen above always hold. And second, we can expand the process of finding linear approximations (for instance, how to approximate non-polynomials like $f(x) = \sin(x)$?). Both ways are accomplished using the same technical device, limits.

Exercises

1. Why should we shift from studying exact functions to studying their approximate actions? Consider the function $f(x) = x^3 - x$. Graph it, and guess the value of x at the local minimum and the local maximum. Next, observe that at the local minimum and the local maximum the tangent line is horizontal — has slope zero. Find the linear approximation at a, set it to zero, and solve.

2. Graph $f(x) = \sin(x)$, being sure that your graph's horizontal scale equals its vertical scale. Estimate the slope of the tangent line at $a = 0$, $\pi/5$, $2\pi/5$, $\ldots$, 2π. Sketch a plot of a versus your estimated slope at a. Make a conjecture about the formula for the slopes.